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An exponential stability criterion for certain nonlinear systems

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SUMMARY
Improved sufficient conditions are derived for the exponential stability of a nonlinear time varying feedback system having a time invariant block G in the forward path and a nonlinear time varying gain $\varphi(.)\,k(t)$ in the feedback path. $\phi(.)$ being an odd monotone nondecreasing function. The resulting bound on $\left(\frac{dk}{dt}/k\right)$ is less restrictive than earlier criteria.

1. Introduction

Consider a feedback system (Fig. 1) governed by the nonlinear differential equation

$$p(D)y+k(t)\phi(q(D)y)=0 \text{ on the interval } [t_0,\infty)\,, \tag{1}$$

where $p(D)=D^n+p_{n-1}D^{n-1}+\ldots+p_0$,

$$q(D)=q_mD^m+q_{m-1}D^{m-1}+\ldots+q_0$$

are constant coefficient differential operators with the order n of $p(D)$ at least one higher than the order m of $q(D)$.

Let $y=x_1$, $x_2=dx_1/dt$, ..., $x_n=dx_{n-1}/dt$; and $\boldsymbol{x}=\mathrm{col}\,[x_1,x_2,\ldots,x_n]$. Then (1) can be written as the vector differential equation

$$\frac{d\boldsymbol{x}}{dt}=A_0\boldsymbol{x}-k(t)\boldsymbol{b}\phi(\boldsymbol{c}'\boldsymbol{x}) \tag{2}$$

$$\triangleq f(\boldsymbol{x},t)$$

(In the figure, $\sigma=q(D)y=\boldsymbol{c}'\boldsymbol{x}$).

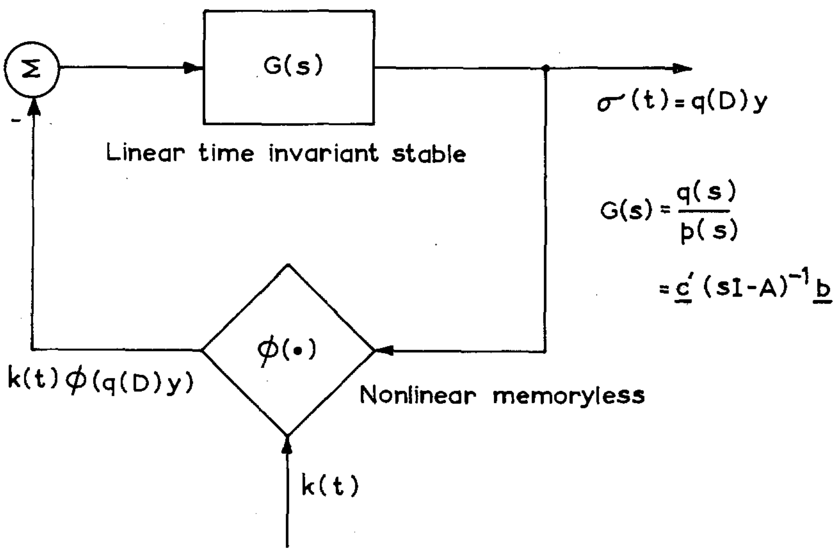

Figure 1. A time varying feedback system.

where A_0 is a stable matrix having the form

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -p_0 & -p_1 & -p_2 & -p_3 & \cdots & -p_{n-1} \end{bmatrix},$$

and $\boldsymbol{b}$, $\boldsymbol{c}$ are vectors given by

$$\boldsymbol{b} = \text{col}[0, 0, \dots, 1]$$
$$\boldsymbol{c} = \text{col}[-q_0, -q_1, \dots, -q_m, \dots, 0].$$

The gain $k(t)$ is assumed to be absolutely continuous on the interval $[t_0, \infty)$; $\phi(.)$ is a real valued function on $(-\infty, +\infty)$ with the following properties: (i) $\phi(0)=0$; (ii) $\phi(.)$ is odd monotone nondecreasing, i.e., $(\sigma_1-\sigma_2)(\phi(\sigma_1)-\phi(\sigma_2)) \geqq 0$ for all σ_1 and σ_2, and $\phi(\sigma) = -\phi(-\sigma)$ for all $\sigma \neq 0$; and there exist constants $q_1, q_2 > 0$ (with $q_1 < q_2$) such that $q_1\sigma^2 \leqq \phi(\sigma)\sigma \leqq q_2\sigma^2$ for all $\sigma \neq 0$. The class of such functions is denoted by $\mathcal{N}$. The equation (1) or (2) with the above specifications is simply designated, for convenience, as system (1). Let $G(s)$ be the transfer function of the forward block, i.e., $G(s)=q(s)/p(s)$.

Assumption: The null solution of (1) is asymptotically stable for every constant function $k(t)=K$ in $[0, \infty)$ when the system is linear, that is, $\phi(\sigma) \equiv \sigma$.

Problem: Find conditions for the exponential stability* of the system (1) for every (absolutely) continuous function $k(t)$ with values in $[0, \infty)$.

The problem of stability of feedback systems with a time varying nonlinearity was initially considered by Zames [1]. Stability conditions in terms of certain positive real functions and a certain upper bound on the rate of variation of $k(t)$ were derived by Narendra and Taylor [2], the author [3a] and many others. More general conditions (in terms of noncausal and causal multiplier functions) are due to the author [3b]. However, these criteria are not necessary for stability, and weaker conditions may be possible.

2. Main results

In the present work, there are two improvements over the existing criteria:

(1) The multiplier is a general positive real function (but with a certain time domain constraint);

(2) The restriction on $\left(\frac{dk}{dt}\Big/k\right)$ is considerably weakened.

However, the main contribution is believed to be Lemma 2 which may be of independent interest. It should be possible, using Lemma 2, to derive frequency power formulas more general than those of Skoog and Willems [4].

The following notation will be used: $\|\boldsymbol{x}\|$ denotes the norm of $\boldsymbol{x}$ where $\|\boldsymbol{x}\|^2 = \boldsymbol{x}'\boldsymbol{x}$ (prime denotes transpose); $\boldsymbol{x}_0$ denotes $\boldsymbol{x}(t_0)$; $\boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ denotes the solution of (2) which takes the value $\boldsymbol{x}_0$ for $t=t_0$.

Definition 1: *The null solution of* (2) *is said to be exponentially stable if there exist positive constants* $\varepsilon_1, \varepsilon_2$, *such that, for all* $t \geqq t_0$,

$$\|\boldsymbol{x}(t; t_0, \boldsymbol{x}_0)\| \leqq \varepsilon_2 \|\boldsymbol{x}_0\| \exp[-\varepsilon_1(t-t_0)].$$

Definition 2: *A complex-valued function* $Z(s)$ *of a complex variable s is called a positive real function of the argument s, if* $Z(s)$ *is real for real values of s, and for* Re $s > 0$, (*where* 'Re' *denotes the real part*) *is analytic and satisfies the inequality*

$$\text{Re}\, Z(s) \geqq 0.$$

* See definition below.

Let $Z(s)=m(s)/n(s)$ where $m(s)$ and $n(s)$ are finite polynomials in s; $\theta(t) \triangleq \left(\frac{dk}{dt}\Big/k\right)$; a superscript + on a time function denotes the positive values of that function. Further, let $Z(s)=1+Z_1(s)$ with $z_1(t)$, the inverse transform of $Z_1(s)$, zero for $t<0$; and finally

$$\delta_s = \sup_{\sigma} \left\{ \int_0^{\sigma} \phi(w)\,dw/\phi(\sigma)\sigma \right\} \tag{3}$$

$$\delta_i = \inf_{\sigma} \left\{ \int_0^{\sigma} \phi(w)\,dw/\phi(\sigma)\sigma \right\} \tag{4}$$

Note that $0<\delta_s \leqq 1$ for $\phi(.)\in\mathcal{N}$.

The main result of the paper is the following.

Theorem 1. *The system* (1) *is exponentially stable if there exists a positive real function* $Z(s) \triangleq m(s)/n(s)=1+Z_1(s)$, *with* $z_1(t)=0$ *for* $t<0$, *such that*

(a) $Z(s-\beta)G(s-\beta)$ *is positive real for some* $\beta \geqq 0$, *and* $m(s-\beta)q(s-\beta)$, $n(s-\beta)p(s-\beta)$ *have no imaginary zeros*;

(b) *for some* $\gamma>0$,

$$\int_0^{\infty} e^{\gamma t}|z_1(t)|\,dt \leqq 1/(1+\delta_s-\delta_i)$$

with δ_s, δ_i *as defined in* (3), (4) *respectively*;

(c) *for* $\alpha = \max \gamma$ *of hypothesis* (*b*),

(i) $\displaystyle\frac{1}{T}\int_{t_0}^{t_0+T} (\theta(\tau)+2\beta-\alpha)^+\,d\tau \leqq M<\infty$ *for all finite* $T>0$;

(ii) $\displaystyle\lim_{T\to\infty}\int_{t_0}^{t_0+T} (\theta(\tau)+2\beta-\alpha)^+\,d\tau \leqq 2\beta - v$ *for some* $v>0$.

The proof of the theorem is somewhat involved and is given at the end, after a series of lemmas which are similar to those found in the author's paper [3c].

3. Some preliminary results

Lemma 1. *If the system of* η *th order described by*

$$p(D)n(D)y+k(t)\phi(q(D)n(D)y=0 \tag{5}$$

is exponentially stable and $n(D)w=0$ *represents an asymptotically stable system, then system* 1 *is also exponentially stable.*

Proof. If $x^*(t)$ is any solution, $\boldsymbol{x}^*(t)$ the corresponding solution vector of (5), then $n(D)\boldsymbol{x}^*(t)$ (or, more precisely, a subvector of $n(D)\boldsymbol{x}^*(t)$) is a solution vector of (1). Further, if $\|\boldsymbol{x}^*(t)\|\to 0$ exponentially, each component of $\boldsymbol{x}^*(t)\to 0$ exponentially, and hence $\|n(D)\boldsymbol{x}^*(t)\|\to 0$ exponentially. Therefore system (1) is exponentially stable. Q.E.D.

Let $\boldsymbol{u}(t)$ denote a state vector of the differential equation

$$m(D)q(D)u+n(D)p(D)u=0\,. \tag{6}$$

Note that the order of (6) is η. From the positive realness of $Z(s-\beta)G(s-\beta)$ for some $\beta \geqq 0$, the system represented by (6) is asymptotically stable. Define

$$r(s)=\{\mathrm{Ev}\, m(s-\beta)q(s-\beta)n(-s-\beta)p(-s-\beta)\}^{(-)}$$

where Ev denotes "even part of" and the superscript $(-)$ stands for the negative spectral factor of the even polynomial inside the brackets. Then we can define the following positive definite function quadratic in $\boldsymbol{u}$:

$$I_1(\boldsymbol{u}, t) = \int_{t(0)}^{t(\boldsymbol{u})} \{[m(D-\beta)q(D-\beta)(u\mathrm{e}^{\beta\tau})][n(D-\beta)p(D-\beta)(u\mathrm{e}^{\beta\tau})] - [r_1(D)(u\mathrm{e}^{\beta\tau})]^2\} d\tau \quad (7)$$

where $u(t)$ is any solution of (6) on $[0, \infty)$, and the integral is a path integral in the state space of (6), parametrised by t. The result is somewhat wellknown and is found, for instance, in [3c].

From the positive definiteness of $I_1(\boldsymbol{u}, t)$, we have $I_1(\boldsymbol{u}, t) \geqq \alpha_0 \|\boldsymbol{u}\, \mathrm{e}^{\beta\tau}\|^2$ for some constant $\alpha_0 > 0$, and hence, if we define

$$V_1(\boldsymbol{u}, t) = \mathrm{e}^{-2\beta t} I_1(\boldsymbol{u}, t) \quad (8)$$

it is easy to conclude that $V_1(\boldsymbol{u}, t) \geqq \alpha_0 \|\boldsymbol{u}\|^2$ and therefore $V_1(\boldsymbol{u}, t)$ is positive definite. Further, by a wellknown property of quadratic forms, there exists a constant δ_1 such that

$$I_1(\boldsymbol{u}, t) \leqq \delta_1 \|\boldsymbol{u}\, \mathrm{e}^{\beta\tau}\|^2$$

from which

$$V_1(\boldsymbol{u}, t) \leqq \delta_1 \|\boldsymbol{u}\|^2 .$$

Lemma 2. *For any function $v_1(.)$, the integral*

$$I_2 = \int_0^t \mathrm{e}^{\alpha\tau} \phi(n(D) v_1(\tau))(m(D) v_1(\tau)) d\tau$$

where α is a nonnegative constant, is nonnegative for $\phi(.) \in \mathcal{N}$ if

$$\int_0^\infty \mathrm{e}^{\alpha t} |z_1(t)| \, dt \leqq 1/(1+\delta_s - \delta_i) \quad (9)$$

with δ_s and δ_i as defined in (3) *and* (4) *respectively.*

Proof. See Appendix 1.

Let $v_1(t) = q(D)u(t)$. Then I_2 becomes a function of $\boldsymbol{u}$ (or its subspace) and t. Define

$$V_2(\boldsymbol{u}, t) = \mathrm{e}^{-\alpha t} \int_0^t \mathrm{e}^{\alpha\tau} \phi(n(D)q(D)u)(m(D)q(D)u) d\tau \quad (10)$$

where u is any solution of (6) and $\alpha > 0$ satisfies inequality (9). It is evident that, if (9) is satisfied, $V_2(\boldsymbol{u}, t)$ is nonnegative. Further, in view of the fact that $\phi(.) \in \mathcal{N}$, there exists a positive constant δ_3 such that $V_2(\boldsymbol{u}, t) \leqq \delta_3 \|\boldsymbol{u}\|^2$ for all nonnegative t.

Finally, let $\zeta(t)$ be a nonnegative (integrable and bounded) function on $[t_0, \infty)$ and $h(t) = \exp[-\int_{t_0}^t \zeta(\tau) d\tau]$. Assume that the integral $\int_{t_0}^t \zeta(\tau) d\tau \leqq M < \infty$ for all t in $[t_0, \infty)$, and

$$0 < \varepsilon \leqq \lim_{t\to\infty} \int_{t_0}^t \zeta(\tau) d\tau \leqq M < \infty .$$

Then $h(t)$ is a bounded positive function. Note that $\left(\frac{dh(t)}{dt} \Big/ h(t)\right) = -\zeta(t)$ which is nonpositive.

Now, let

$$V(\boldsymbol{u}, t) = h(t) \{V_1(\boldsymbol{u}, t) + k(t) V_2(\boldsymbol{u}, t)\} \quad (11)$$

where $h(t)$, $V_1(\boldsymbol{u}, t)$ and $V_2(\boldsymbol{u}, t)$ are as defined above. By virtue of the boundedness of $k(t)$ and $h(t)$, and the properties of $V_1(\boldsymbol{u}, t)$, $V_2(\boldsymbol{u}, t)$ specified above, the following inequality holds

$$\gamma_0 \|\boldsymbol{u}\|^2 \leqq V(\boldsymbol{u}, t) \leqq \gamma_1 \|\boldsymbol{u}\|^2 \quad (12)$$

where γ_0 and γ_1 are positive constants.

Based on the above auxiliary results, the following lemma can be proved. The method of proving it is quite straightforward and simple, and is hence omitted. (See, for instance, [3c]).

Lemma 3. *Let $V(\boldsymbol{u}, t)$ be as defined above. Its time derivative is given by*

$$\frac{dV(\boldsymbol{u}, t)}{dt} = h(t)\{-2\beta V_1(\boldsymbol{u}, t) + [m(D)q(D)u][n(D)p(D)u] - [r(D+\beta)u]^2 + k(t)\phi(n(D)q(D)u)(m(D)q(D)u) + + [\theta(t)-\alpha]k(t)V_2(\boldsymbol{u}, t)\} - \zeta(t)V(\boldsymbol{u}, t) \tag{13}$$

which along the trajectories of (5) *assumes the value* (*obtained by substituting $x^*(t)$ for $u(t)$ and $\boldsymbol{x}^*(t)$ for $\boldsymbol{u}(t)$ in* (13))

$$\left.\frac{dV(\boldsymbol{x}^*, t)}{dt}\right|_{(5)} = h(t)\{-2\beta V_1(\boldsymbol{x}^*, t) - [r(D+\beta)x^*]^2 + [\theta(t)-\alpha]k(t)V_2(\boldsymbol{x}^*, t)\} - \zeta(t)V(\boldsymbol{x}^*, t)$$

and this satisfies the inequality, for some $\delta_0 \geqq 0$,

$$\left.\frac{dV(\boldsymbol{x}^*, t)}{dt}\right|_{(5)} \leqq \sup_{t\geqq 0}\{[-2\beta-\delta_0, \theta(t)-\alpha] - \zeta(t)\} V(\boldsymbol{x}_1^*(t)\,. \tag{14}$$

We now give the last preliminary result (due, in essence, to Corduneanu [5]) on which the proof of Theorem 1 hinges.

Lemma 4. (*Corduneanu* [5]). *If there exist a positive definite and decrescent form $v(\boldsymbol{x}, t) = \boldsymbol{x}' P(t)\boldsymbol{x}$ and a real valued function $\lambda(t)$ on $[t_0, \infty)$ such that the derivative of $v(\boldsymbol{x}, t)$ along the solutions of* (2) *satisfies the inequality*

$$\left.\frac{dv}{dt}\right|_{(2)} \leqq -\lambda(t)v$$

then there exists a positive constant μ_0 such that the solutions of (2) *satisfy the inequality*

$$\|\boldsymbol{x}(t)\| \leqq \mu_0 \|\boldsymbol{x}(t_0)\| \exp\left(-\tfrac{1}{2}\int_{t_0}^{t} \lambda(\tau)\,d\tau\right).$$

Proof: See Appendix 2.

Corollary. If $T^{-1}\int_0^{t_0+T}(-\lambda(\tau))\,d\tau \leqq -\nu$ for some positive constant ν and for all $T > 0$, then $\|\boldsymbol{x}(t)\| \leqq \mu_0\|\boldsymbol{x}(t_0)\| \exp(-\nu(t-t_0)/2)$, and the system (2) is exponentially stable.
The proof of the main result follows.

Proof of Theorem 1. As a Lyapunov–Corduneanu function candidate for (5), choose

$$V(\boldsymbol{x}^*, t) = h(t)\{V_1(\boldsymbol{x}^*, t) + k(t)V_2(\boldsymbol{x}^*, t)\}$$

where $V_1(\boldsymbol{x}, t)$, $V_2(\boldsymbol{x}^*, t)$ are defined by (8) and (10) respectively, and $h(t)$ is defined prior to Lemma 3. $V(\boldsymbol{x}^*, t)$ is positive definite, radially unbounded, has continuous partial derivatives, and satisfies decrescent conditions by virtue of the boundedness of $k(t)$ and $h(t)$. (See inequality (12)). Its time-derivative along the solutions of (5) satisfies inequality (14). Invoking Corollary of Lemma 4, we conclude that hypothesis (c) implies exponential stability. Q.E.D.

Remarks: (1) From the proof of Lemma 2, it is obvious that Theorem 1 holds when $\phi(.)$ belongs to a class of monotone nondecreasing functions which do not possess the odd property, if $z_1(t) \leqq 0$. (2) The present condition on $\theta(t)$ is an average condition and permits large positive variations of a shifted $\theta(t)$ over a finite interval; also note that the negative lobes of the shifted

$\theta(t)$ do not enter into the picture. (3) A geometric interpretation of Theorem 1 is desirable in view of its dependence on a multiplier function whose construction is not given. The method of Freedman [6] can be suitably modified but lacks the elegance and simplicity of Nyquist-type criteria.

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Appendix 1

Proof of Lemma 2. Let $n(D)v_1(\tau)=\sigma_1(\tau)$. We have

$$I_2 = \int_0^t e^{\alpha\tau}\phi(\sigma_1(\tau))\sigma_1(\tau)d\tau + \int_0^t e^{\alpha\tau}\phi(\sigma_1(\tau))\left(\int_0^\infty z_1(\tau')\sigma_1(\tau-\tau')d\tau'\right)d\tau\,. \tag{15}$$

Assuming that the order of integration in the last integral of (15) can be interchanged, we get (after some minor manipulations)

$$I_2 = \int_0^t e^{\alpha\tau}\phi(\sigma_1(\tau))\sigma_1(\tau)d\tau + \int_0^\infty z_1(\tau')e^{\alpha\tau'}\left[\int_0^t \phi(\sigma_1(\tau))\sigma_1(\tau-\tau')e^{\alpha(\tau-\tau')}d\tau\right]d\tau' \tag{16}$$

From the monotone property of $\phi(.)$ we have

$$\phi(y_1)(y_1-y_2)\geqq\Phi(y_1)-\Phi(y_2) \text{ for all } y_1 \text{ and } y_2\,. \tag{17}$$

where $\Phi(y_1)=\int_0^{y_1}\phi(\sigma_1)d\sigma_1$.

Define $y_1=\sigma_1(\tau)$ and $y_2=\sigma_1(\tau-\tau')e^{-\alpha\tau'}$. Using (17), we can write the following inequality

$$\int_0^t e^{\alpha\tau}\phi(\sigma_1(\tau))(\sigma_1(\tau)-\sigma_1(\tau-\tau')e^{-\alpha\tau'})d\tau \geqq \int_0^t e^{\alpha\tau}\Phi(\sigma_1(\tau))d\tau - \int_0^t e^{\alpha\tau}\Phi(\sigma_1(\tau-\tau')e^{-\alpha\tau'})d\tau\,. \tag{18}$$

The last integral of (18) can be rewritten by changing the variable of integration to $\tau_1=\tau-\tau'$:

$$\int_0^t e^{\alpha\tau}\Phi(\sigma_1(\tau-\tau')e^{-\alpha\tau'})d\tau = \int_{-\tau'}^{t-\tau'} e^{\alpha(\tau_1+\tau')}\Phi(\sigma_1(\tau_1)e^{-\alpha\tau'})d\tau_1$$

But $\Phi(\sigma_1(\tau_1)e^{-\alpha\tau'})\leqq\delta_s\phi(\sigma_1 e^{-\alpha\tau'})\sigma_1(\tau_1)e^{-\alpha\tau'}$ from inequality (3)), and $\Phi(\sigma_1(\tau))\geqq \delta_i\phi(\sigma_1(\tau))\sigma_1(\tau)$ (from inequality (4)).

Therefore, noting that $\tau'\geqq 0$ and that $\sigma_1(\tau_1)=0$ for $\tau_1<0$, the right-hand side of inequality (18) is greater than or equal to

$$\delta_i\int_0^t e^{\alpha\tau}\phi(\sigma_1(\tau))\sigma_1(\tau)d\tau-\delta_s\int_0^t e^{\alpha\tau}\phi(\sigma_1(\tau)e^{-\alpha\tau'})\sigma_1(\tau)d\tau\,. \tag{19}$$

Once again, from the monotonicity of $\phi(.)$, we have

$$\phi(\sigma_1(\tau)e^{-\alpha\tau'})\sigma_1(\tau)\leqq\phi(\sigma_1(\tau))\sigma_1(\tau)$$

which, when used in (19) and subsequently in (18), gives

$$\int_0^t e^{\alpha\tau}\phi(\sigma_1(\tau))(\sigma_1(\tau-\tau')e^{-\alpha\tau'})d\tau\leqq(1+\delta_s-\delta_i)\int_0^t e^{\alpha\tau}\phi(\sigma_1(\tau))\sigma_1(\tau)d\tau\,. \tag{20}$$

Consequently, from (16) we conclude that I_2 is nonnegative if $z_1(\tau') \leqq 0$ and inequality (9) is satisfied.

But $\phi(.)$ also has the property of being odd, i.e., $\phi(\sigma) = -\phi(-\sigma)$ for all $\sigma \neq 0$. Now write the monotone inequality (17) for $(-y_2)$ and carry through the above calculation to infer

$$\left| \int_0^t e^{\alpha\tau} \phi(\sigma_1(\tau))(\sigma_1(\tau-\tau')e^{-\alpha\tau'})d\tau \right| \leqq (1+\delta_s-\delta_i) \int_0^t e^{\alpha\tau}\phi(\sigma_1(\tau))\sigma_1(\tau)d\tau \tag{21}$$

which in association with (16) proves the lemma. Q.E.D.

Appendix 2

Proof of Lemma 4. Because $v(x, t)$ is positive definite and decrescent, there exist positive constants α_1 and α_2 such that

$$\alpha_1 \|x\|^2 \leqq x'P(t)x \leqq \alpha_2 \|x\|^2 .$$

Integration of the inequality

$$\left.\frac{dv}{dt}\right|_{(2)} \leqq -\lambda(t)v$$

gives

$$v(x, t) \leqq v(x_0, t) \exp\left(-\int_{t_0}^t \lambda(\tau)d\tau\right).$$

Consequently,

$$\alpha_1 \|x\|^2 \leqq v(x, t) \leqq v(x_0, t) \exp\left(-\int_{t_0}^t \lambda(\tau)d\tau\right) \leqq \alpha_2 \|x_0\|^2 \exp\left(-\int_{t_0}^t \lambda(\tau)d\tau\right)$$

from which

$$\|x\|^2 \leqq \frac{\alpha_2}{\alpha_1} \|x_0\|^2 \exp\left(-\int_{t_0}^t \lambda(\tau)d\tau\right).$$

Therefore

$$\|x\| \leqq \mu_0 \|x_0\| \exp\left(-\tfrac{1}{2}\int_{t_0}^t \lambda(\tau)d\tau\right)$$

where $\mu_0 = (\alpha_2/\alpha_1)^{\frac{1}{2}}$, and the lemma is proved. Q.E.D.

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